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The Constitutive Equations of Continuum Creep Damage Mechanics [and Discussion]

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The constitutive equations of continuum creep damage mechanics

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[Plates 1 and 2]

When metals are subjected to stress at temperatures in excess of about one third of their melting temperature, it is found that they deform continuously and that they can sustain the stress for a limited time referred to as the rupture life. Constitutive equations are proposed which describe the macroscopic behaviour of metals when subjected to multiaxial states of stress. In these constitutive equations it is necessary to introduce the concept of an internal state parameter ω referred to as the damage parameter. It is shown that under certain loading conditions this damage parameter is a normalized form of the actual physical damage which is defined as the integral of nucleation and growth rates of voids which grow on the grain boundary. The constitutive equations are used to determine the characteristics of certain load-bearing components operating at temperature. It is shown that the reference rupture stress and the isochronous rupture surface are the material properties which are responsible for component performance.

1. Introduction

When load-bearing metallic components operate at temperatures in excess of approximately one third of the melting temperature T_m consideration must be given to the effects of creep deformations and rupture. In some components such as those in use in the nuclear industry limitations are placed on the amount of creep strain which may be accumulated. In other components such as those occurring in the process industries less consideration need be given to deformation and the life is only terminated when rupture takes place and some form of leakage becomes evident. Rupture can be the consequence of thinning induced by large strains but it can also occur at small strains as the result of growth of damage in the metal. Metallographic inspection reveals that the damage normally occurs in the form of fissures and voids which coincide with grain boundaries. In some metals such as copper it is easy to observe the damage by visual inspection, while in others such as aluminium alloys the damage is difficult to observe in the optical microscope. In this paper attention is given to the means of describing metallic creep rupture behaviour by suitable constitutive equations which can be used to predict the creep life of components with complex geometries. This branch of mechanics is often referred to as *continuum creep damage mechanics* because damage is observed to be smoothly distributed, in contrast to classical fracture when damage appears to propagate along a well-defined line.

An example illustrating the continuous distribution of creep rupture damage is that of a plate penetrated by a circular hole and subjected to constant load. The damage in a copper plate tested at 250 °C is shown in figure 1, plate 1, from which it can be seen that the damage varies continuously with the greatest concentration of damage at the edge of the hole. This test was conducted by Leckie & Hayhurst (1974) who also observed that the rupture life of the plate was dictated by the average tensile stress at the minimum section of the plate. For the particular plate geometry the maximum elastic stress is 2.2σ and the maximum steady state creep stress is 1.4σ where σ is the average stress at the minimum cross section. Predictions of life based on these two values would therefore give values which are much shorter than those observed experimentally. The implication is that the effect of stress redistribution is sufficiently large for the life of the plate to be dictated by the average stress at the minimum section. The initial stress concentrations appear to have no effect on the rupture life of the plate. In order to verify this observation further a plate of identical size was penetrated by a slit of length equal to the diameter of the circular hole. It was found that stress concentration effects could be neglected with the rupture life dependent on the value of the average stress at the minimum section.

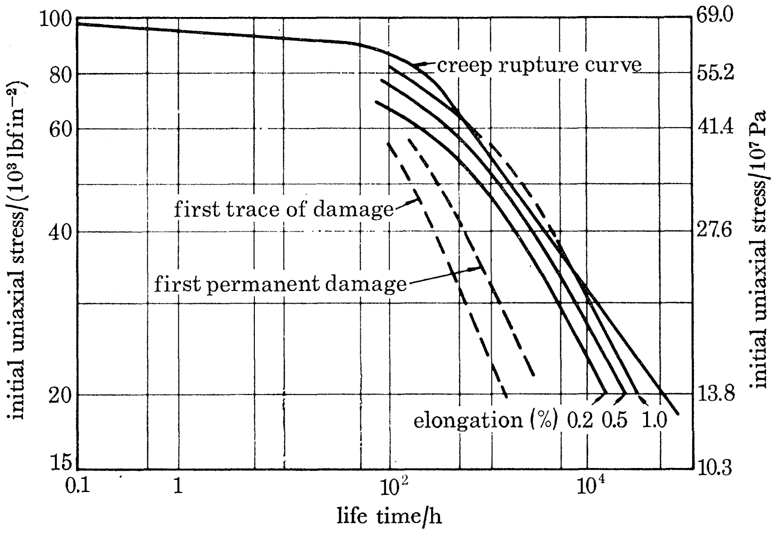

FIGURE 2. Uniaxial stress/time to rupture data for a low alloy steel at 500 °C (1 lbf in^{-2} ≈ 69 kPa).

The principal interest of engineers is to predict the magnitude of deformations and rupture time of load-bearing components, and little attention is given to describing the physical processes occurring in the material which is the special interest of the metallurgist. It is not surprising therefore that the procedures developed to suit the needs of the two groups are apparently at odds. An attempt is made in this paper to illustrate that the two approaches are in many respects quite similar and that only modest modifications are required to bring them into agreement. It is then shown how the proposed equations can be used in conjunction with continuum mechanics procedures to predict component behaviour. It is convenient to relate this behaviour to available uniaxial creep rupture data by means of the so-called *reference rupture stress* which, it is suggested, is the dominant material parameter to be used in design against failure by creep damage.

2. Creep rupture data

The great majority of engineering creep data is obtained from constant uniaxial stress tests. Normally the rupture time t_R is plotted against the applied stress σ with results similar to that shown in figure 2. Sometimes contours of constant strain are also plotted as shown in figure 2 but this information is much less common. Referring to figure 2 it can be seen that when $\lg t_R$ is plotted against $\lg \sigma$ the resulting graph consists of two straight lines with a transition curve between. The failure at the higher values of stress is normally the result of necking induced by large creep strains. At the lower stress levels the failure is associated with material damage

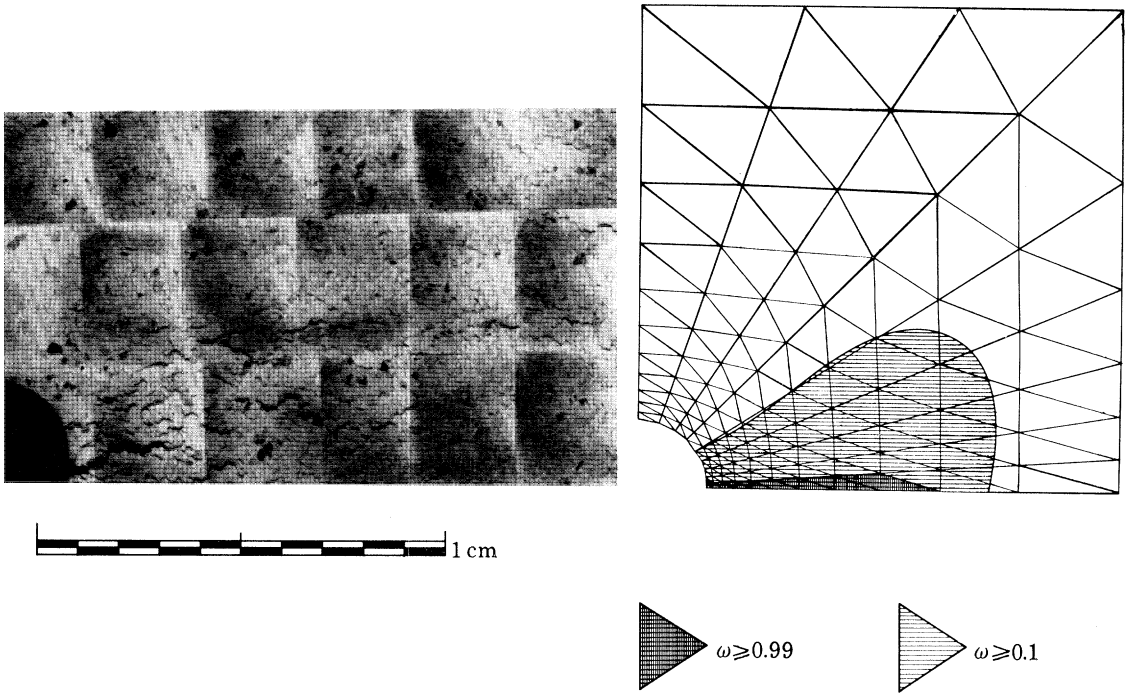

FIGURE 1. Rupture of copper tension panel with hole.

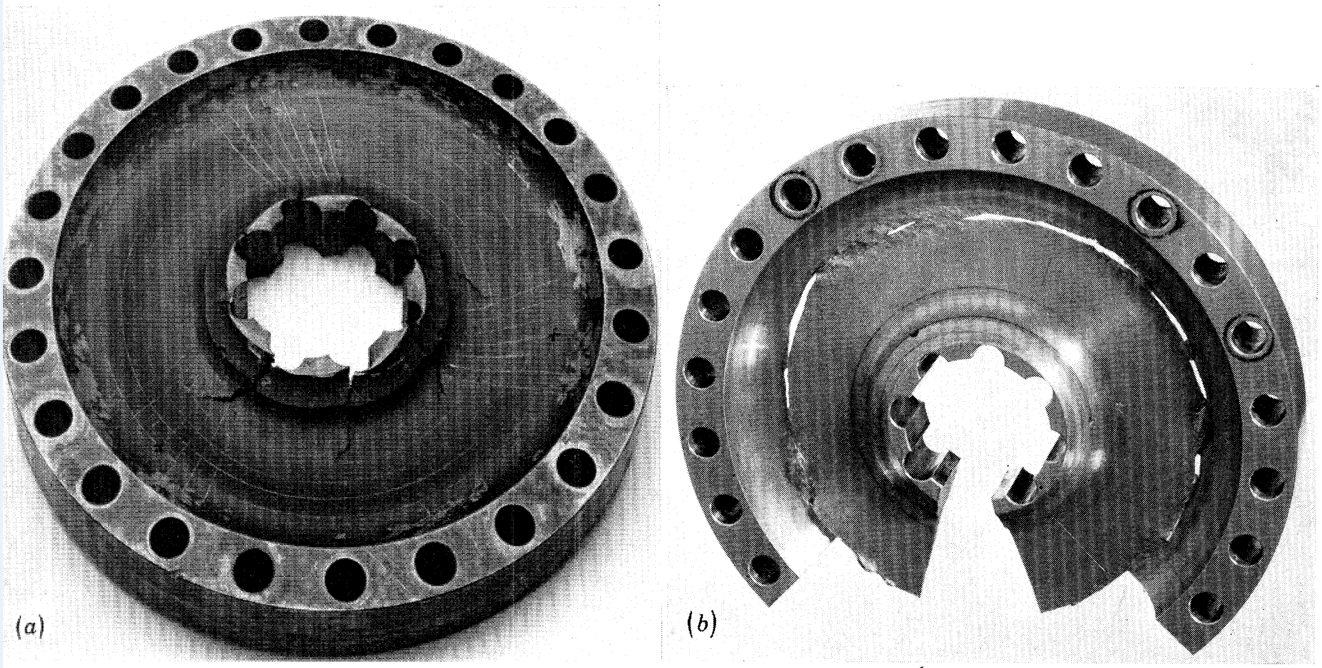

FIGURE 4. Andrade disk (*a*) copper (*b*) aluminium.

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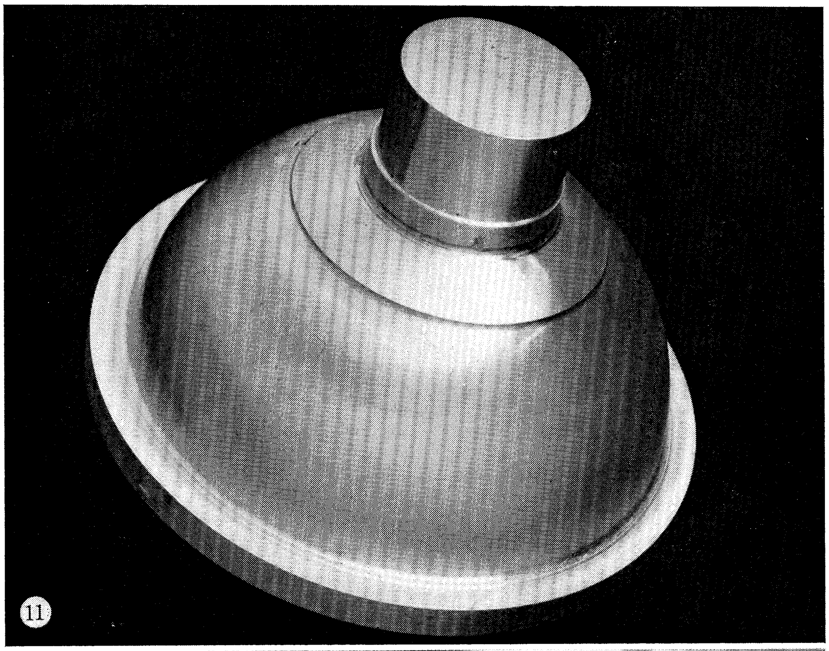

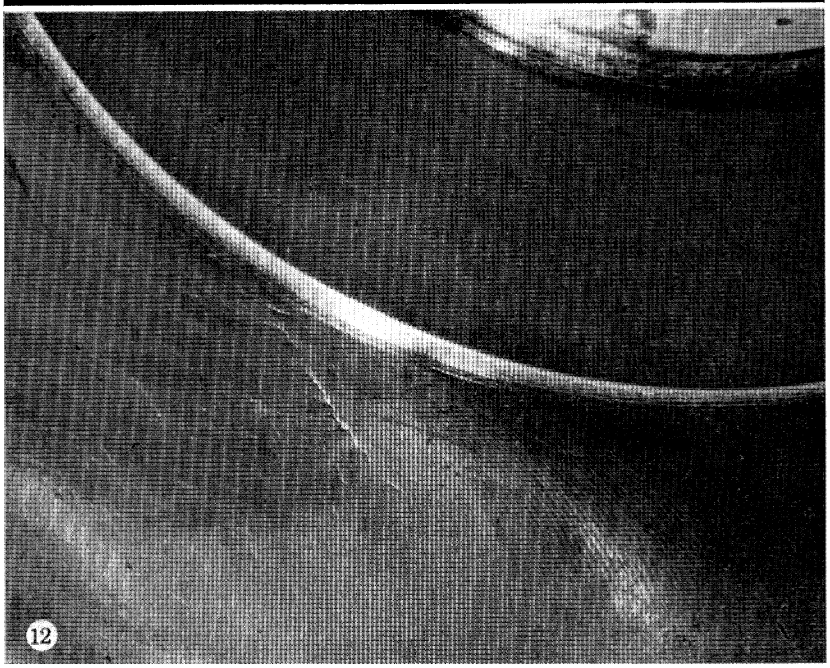

FIGURE 11. Balanced reinforced nozzle.
FIGURE 12. Rupture failure of membrane.

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when the strains at rupture are much smaller. In practice the operating stress levels are normally on the low side being below those occurring in the transition region of the rupture curve. Consequently failure is normally associated with internal damage. If t_0 is the rupture time associated with the applied stress σ_0 then the creep rupture time t_R for the applied stress σ is given by

$$t_R/t_0 = (\sigma_0/\sigma)^{\nu}, \tag{2.1}$$

where ν is a material constant equal in magnitude to the slope of the rupture curve $g\sigma$ against $\lg t_R$.

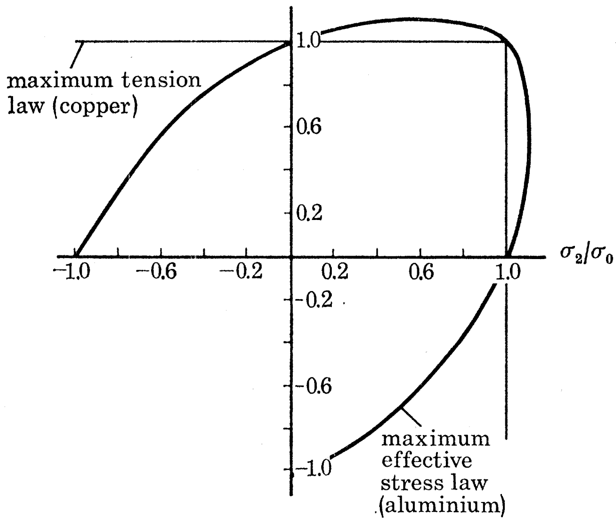

FIGURE 3. Plane stress isochronous rupture loci for copper and aluminium alloys.

In the absence of damage the steady state creep rate $\dot{\epsilon}_s$ can usually be expressed in the form

$$\dot{\epsilon}_s/\dot{\epsilon}_0 = (\sigma/\sigma_0)^n. \tag{2.2}$$

In this equation $\dot{\epsilon}_0$ is the steady state creep rate corresponding to the applied stress σ_0 and n is a material constant. In many materials the value of n is constant except for high values of stress when the value can increase.

Since most engineering components operate under triaxial stress states there is a particular desire to understand how materials behave in these circumstances. The experimental studies of Johnson, Henderson & Khan (1962) show that the magnitudes of the strain rate is dictated by the Mises or effective stress ($\bar{\sigma}$) criterion where

$$\bar{\sigma} = 2^{-\frac{1}{2}}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]^{\frac{1}{2}}. \tag{2.3}$$

If the effective strain rate $\bar{\dot{\epsilon}}_s$ is defined according to the condition

$$\bar{\dot{\epsilon}}_s = 2^{\frac{1}{2}}\times\tfrac{1}{3}[(\dot{\epsilon}_1-\dot{\epsilon}_2)^2+(\dot{\epsilon}_2-\dot{\epsilon}_3)^2+(\dot{\epsilon}_3-\dot{\epsilon}_1)^2]^{\frac{1}{2}}, \tag{2.4}$$

then the general form of equation (2.2) becomes

$$\bar{\dot{\epsilon}}_s/\dot{\epsilon}_0 = (\bar{\sigma}/\sigma_0)^n. \tag{2.5}$$

In the above equations the subscripts 1, 2, 3 refer to the principal stresses and strain rates with σ_1 being the maximum value.

The effect of multiaxial states of stress on rupture are normally expressed in terms of *isochronous surfaces*. These are surfaces in stress space for which rupture times are constant. As a result of experiments on tubes subjected to torsion and tension Johnson *et al.* (1962) concluded that the rupture life of copper tested at 250 °C is controlled by the value of maximum stress σ_1 while that of an aluminium alloy tested at 200 °C is controlled by the value of effective stress $\overline{\sigma}$. These criteria for biaxial states of stress are illustrated in figure 3 and Johnson *et al.* suggested that the two criteria represent two extremes of material behaviour. Hayhurst (1972), after an extensive review of available data, concluded that isochronous surfaces for rupture time t_0 can be represented by the equation

$$\Delta(\sigma_{ij}/\sigma_0) = \left[\alpha\frac{\sigma_1}{\sigma_0}+\beta\frac{\overline{\sigma}}{\sigma_0}+\gamma\frac{J_1}{\sigma_0}\right] = 1. \tag{2.6a}$$

In this expression α, β, γ are constants with $\alpha+\beta+\gamma = 1$, σ_1 is the maximum stress, $\overline{\sigma}$ the effective stress and $J_1(=\sigma_1+\sigma_2+\sigma_3)$ is the first stress invariant. When $\beta = \gamma = 0$ the rupture criterion is maximum stress dependent, and when $\alpha = \gamma = 0$ it is effective stress dependent. Both of these rupture criteria pass through the point (1.1) on the plane stress loci shown in figure 3. The constant γ is introduced to describe the reduction in the value of stress sometimes observed when metals are subjected to equal biaxial states of stress. It appears however, that γ is usually small (Hayhurst 1972) and when neglected (2.6*a*) simplifies to the form suggested by Sdobyrev (1958):

$$\Delta(\sigma_{ij}/\sigma_0) = [\alpha(\sigma_1/\sigma_0)+(1-\alpha)\,(\overline{\sigma}/\sigma_0)] = 1. \tag{2.6b}$$

A survey of the results of tests designed to investigate the effect of variable stress histories on rupture life has been carried out by Penny & Marriott (1971). The most common test consists of the application of cyclic stress histories and it is then found that the lives are predicted quite satisfactorily by the time summation rule

$$\Sigma t_i/t_R^i = 1, \tag{2.7}$$

where t_R^i is the rupture time for a constant stress test at stress σ_i, and t_i is actual time of application of the stress σ_i. Storåkers (1969) has shown that this result can be less accurate for other forms of stress history and a reliable form of law valid for any stress history remains to be found.

The material information described in this section is the most that the designer can normally expect. A more common experience is that only the results of short-term tests are available and time extrapolation methods must be used. A variety of extrapolation techniques are available but the most scientific and reliable appear to be the so-called 'creep rupture maps' proposed by Ashby & Raj (1975). In these maps the regions in which various creep damage mechanisms are dominant are defined by boundaries of stress and temperature. A given working stress and temperature will fall within a region for which a particular mechanism is dominant and it is important to limit the use of extrapolation techniques to that particular region.

3. Approximate description of component behaviour

On the basis of available information it is now possible to speculate on how different components might behave. The basis of the speculations are:

(*a*) Stress redistribution occurs so that the rupture life of a component is based on average rather than maximum stresses, the supporting evidence being the results of the experiments described in § 1.

(*b*) Creep strains are proportional to the effective stress $\overline{\sigma}$ to some power n.

(*c*) Rupture life of the material is inversely proportional to $\Delta(\sigma_{ij}/\sigma_0)$ (equation 2.6) to some power ν. The equation $\Delta(\sigma_{ij}/\sigma_0) = 1$ defines the isochronous surface for failure time t_0.

After the suggestion of Johnson *et al.* (1962) that the rupture life of copper is maximum stress (σ_1) dependent while that of aluminium is effective stress ($\overline{\sigma}$) dependent, it appears sensible to investigate the properties of different load bearing components made from these different materials. Two simple components have been selected to illustrate the difference in behaviour which might be expected. These are the Andrade disk (figure 4, plate 1) and the Bridgman notch (figure 5).

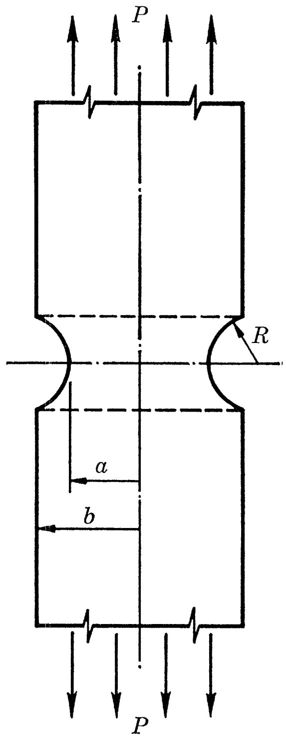

FIGURE 5. The Bridgman notch.

The copper specimens were tested at 250 °C and for copper it is found that in equations (2.1) and (2.5) $n \approx \nu \approx 6$ and that the strain at failure is ϵ_0^R approximately 7 %. For the aluminium alloy HF9 tested at 150 °C the corresponding constants are $n \approx \nu \approx 10$ and the strain at failure ϵ_0^R is 1.5 %. Strain at failure in both of these materials shows small variations with stress, the strain decreasing with decreasing stress. However, it might be noted that there is considerable difference between the failure strains of the copper and aluminium.

(*a*) *The Andrade & Jolliffe disk* (1952)

This is a circular disk which is profiled so that when the disk is subjected to torque about the axis of symmetry the shear stress τ is everywhere constant (figure 4). For this field the maximum stress is $\sigma_1 = \tau$ and the effective stress $\overline{\sigma} = \tau\sqrt{3}$ so that the ratio $\sigma_1/\overline{\sigma} = 1/\sqrt{3} = 0.577$.

Since the rupture life of copper is maximum stress (σ_1) dependent then the stress required in the disk to give a rupture time t_0 is $\sigma_1 = \tau = \sigma_0$. Compared with the corresponding uniaxial test with rupture life t_0 the effective stress is increased by a factor $\sqrt{3}$ and consequently the strain to failure is $\epsilon_0^R(\sqrt{3})^n = 27.0\epsilon_0^R$. Hence the strain at failure of the disk is many times greater than the failure strain in the uniaxial specimen for the same rupture time.

The rupture life of the aluminium alloy is effective stress ($\bar{\sigma}$) dependent and the shear stress required to give a rupture time t_0 is given by the equation $\bar{\sigma} = \tau\surd 3 = \sigma_0$ or $\tau = \sigma_0/\surd 3$. Since the effective stress $\bar{\sigma} = \sigma_0$ in the corresponding uniaxial test with rupture time t_0, the strain at failure in the disk is equal to that in the corresponding uniaxial test.

For ease of comparison the shear stresses required to give a rupture time t_0 together with the strains at failure are given in table 1. In this test therefore the copper disk should be considerably stronger than the aluminium disk when expressed in terms of the normalizing uniaxial stress σ_0. If test results of shear stress τ against failure time are plotted on uniaxial rupture data the results should have the form shown in figure 6. The strain at failure in the copper disk should be many times greater than the uniaxial strain while that in the aluminium disk should be the same.

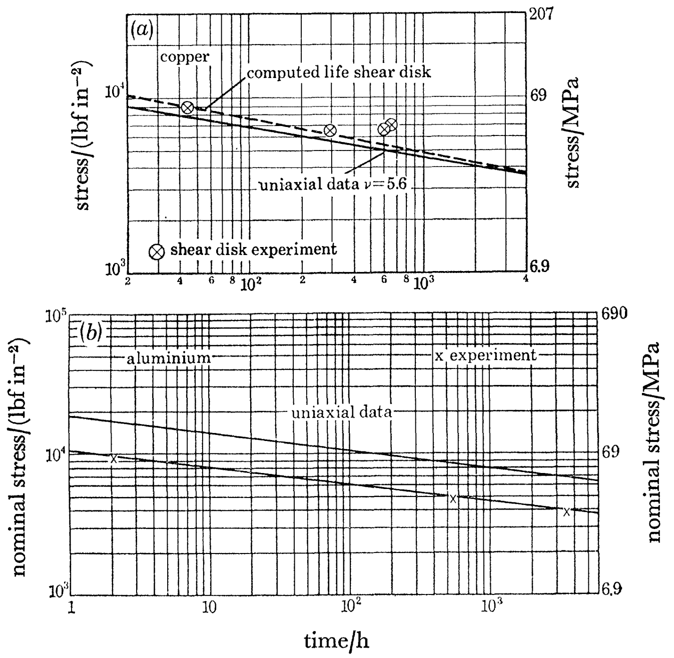

FIGURE 6. Comparison of rupture times for shear disk and uniaxial tension specimens (*a*) copper and (*b*) aluminium.

TABLE 1. STRESSES AND STRAINS FOR RUPTURE TIME t_0

component	max. stress (σ_1)	effective stress ($\bar{\sigma}$)	$\sigma_1/\bar{\sigma}$	failure stress for rupture time t_0: copper	aluminium	failure strain: copper	aluminium
uniaxial	σ_0	σ_0	1.00	σ_0	σ_0	ϵ_0	ϵ_0
Andrade disk	τ	$\tau\surd 3$	0.577	$\tau = \sigma_0$	$\tau = \sigma_0/\surd 3$	$27\epsilon_0$	ϵ_0
Bridgman notch	σ_N	$\sigma_N/1.33$	1.33	$\sigma_N = \sigma_0$	$\sigma_N = 1.33\sigma_0$	$0.18\epsilon_0$	ϵ_0

Experiments have been made on copper and aluminium disks by Hayhurst & Storåkers (1976). The experimental results are in good agreement with the theoretical predictions (figure 6). The large strains observed in the copper disks are illustrated in figure 4 which shows the distorted form of lines which were initially radial.

(b) *The Bridgman notch* (1952)

The introduction of a notch into a cylindrical specimen of the type analysed by Bridgman for perfectly plastic materials can be used to create multiaxial states of stress. For the geometry shown in figure 5 the average normal stress is $\sigma_N = P/\pi a^2$ where P is the applied load and a the radius of the specimen at the minimum cross section. The average maximum stress is then $\sigma_1 = \sigma_N$ while the effective stress is $\overline{\sigma} = \sigma_N/1.33$. Hence the ratio $\sigma_1/\overline{\sigma} = 1.33$, which is approximately twice the value occurring in the Andrade disk.

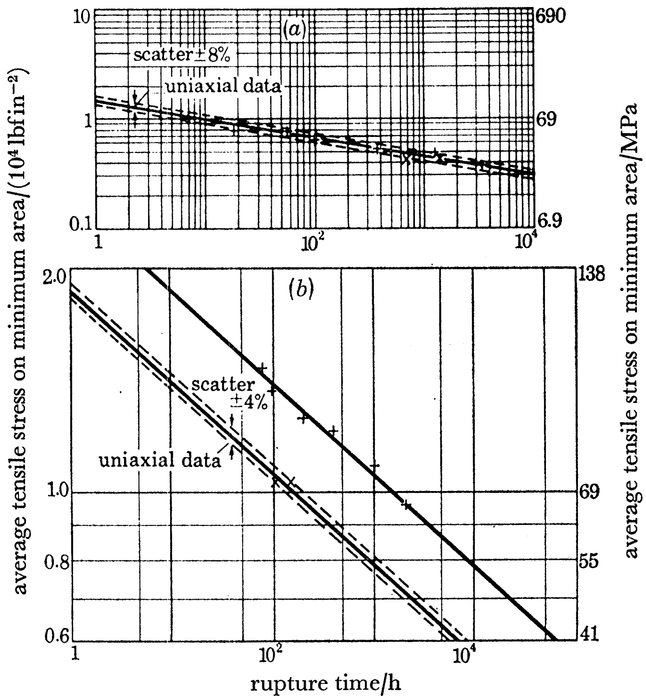

FIGURE 7. Comparison of rupture times for the Bridgman notch and uniaxial tension specimens (a) copper (b) aluminium.

In a copper notch the failure time is achieved with a normal stress $\sigma_N = \sigma_1 = \sigma_0$. The effective stress is $\overline{\sigma} = \sigma_0/1.33$ and consequently the strain at failure should be $\epsilon_0(1/1.33)^6 = 0.18\epsilon_0$. In an aluminium notch the time to failure t_0 should be achieved when $\overline{\sigma} = \sigma_N/1.33 = \sigma_0$, or $\sigma_N = 1.33\sigma_0$. Since the effective stress is equal to that in a uniaxial test the strain at failure should be ϵ_0 (table 1). When the rupture life stress σ_N results are plotted on uniaxial data the results should have the form illustrated in figure 7.

Tests on Bridgman notches have been performed by Leckie & Hayhurst (1974). The results of these tests are in agreement with the predictions of theory. The aluminium specimens show the expected strengthening when compared with uniaxial data, and the failure strains in the copper notches are approximately 2 % which is considerably less than the 7 % observed in uniaxial tests (figure 7).

The predictions and experimental observations illustrate that the behaviour of components is strongly dependent on the combined effect of the multiaxial stress state and the isochronous surface. In the Andrade disk it is the copper which is stronger than aluminium while in the

Bridgman notch the position is reversed. In aluminium the strain at failure is equal to the strain at failure in the uniaxial test. However, in copper the strain to failure can differ greatly in the different components. This result indicates that a commonly held view that the strain at failure is constant may not always be justified.

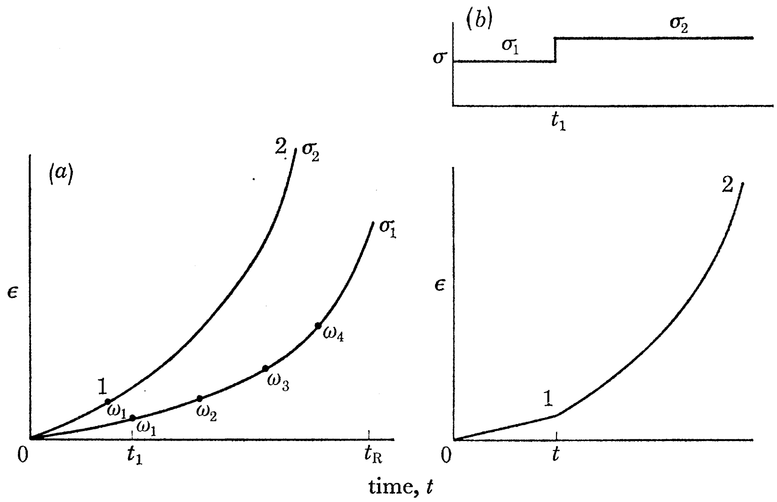

FIGURE 8. State variable tests.

4. Constitutive equations using the state variable description

The predictions made in the previous section were based on the assumption that the rupture life of a component is based on the average values of the normal and effective stresses. The average values were determined using stress distributions available from plasticity theory. The experimental results indicate that procedures along the suggested lines may be valid, but a precise theoretical justification is still lacking. Furthermore, engineers are anxious to study the growth of strains and of damage fronts in components since local failure might result in leakage, while general failure is not imminent. The engineering approach is to develop constitutive equations which describe the macroscopic behaviour of the material. It is the constitutive equations in conjunction with the laws of continuum mechanics which determines component behaviour. In the case of creep rupture, attempts will be made to calculate the time at which cracks first appear, the growth of damage fronts and the time for complete rupture. In order to describe the macroscopic behaviour of the materials it is necessary to introduce internal state variables which, in some sense, are a measure of the physical state of the material. Since engineers do not attempt to give a precise physical description of the state variable this approach, on the face of it, is at variance with the approach of the metallurgist who wishes to give an accurate description of the mechanisms which take place. It will be suggested in the text, however, that the state variable method and the methods used by metallurgists to describe physical phenomena are, in some circumstances, almost identical. In these circumstances it would appear that the state variable description can be used with confidence.

(a) *The state variable description for uniaxial stress*

If, for convenience of discussion, the primary portion of the creep curve is neglected then the strain/time curve in a constant stress test will have the form shown in figure 8*a*. The creep rate increases from the initial steady state value as damage causes an advance into the tertiary portion of the curve. The steady-state creep rate can be expressed as a function of the applied stress σ alone. In order to account for the increase in strain rate during a constant stress test it is necessary to introduce a new variable into the strain equation. Since the increase in strain rate is the result of a damage process the variable ω introduced is referred to as the damage state variable. The strain rate equation then takes the form

$$d\epsilon/dt = f(\sigma, \omega), \tag{4.1a}$$

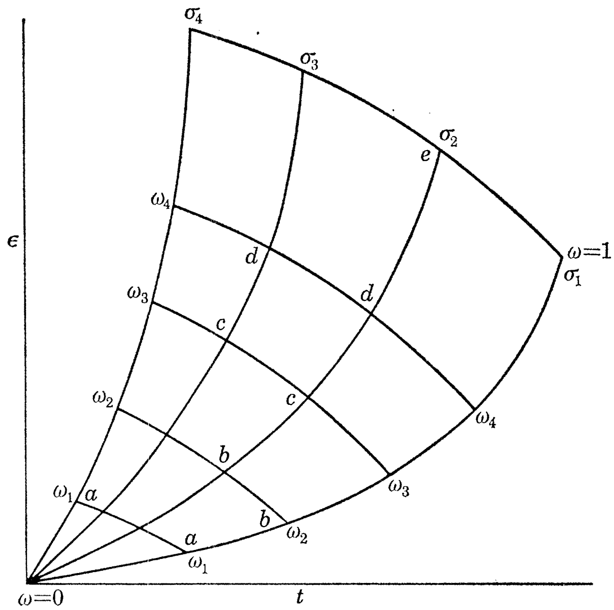

FIGURE 9. Constant damage contours.

where f is a function which has yet to be defined. However, in the undamaged state when $\omega = 0$ the equation must reduce to the form observed for steady state conditions. To define the strain rate it is necessary that the value of ω as well as the stress σ be known and consequently an equation must be introduced which defines the growth of the damage state variable. Assuming that the damage rate depends both on the current state of stress and damage gives the equation

$$d\omega/dt = g(\sigma, \omega). \tag{4.1b}$$

The problem is how to find the functions f and g in a systematic manner. This may be achieved by following a testing procedure suggested by Leckie & Hayhurst (1975) which makes use of the results of tests with step changes in stress. Neglecting the primary portion of the creep curve of the strain/time graphs for two constant stress tests carried out at stresses σ_1 and σ_2 would have the form shown in figure 8a. Suppose in the constant σ_1 test that ω varies between zero at time $t = 0$ and unity at the rupture time t_R. Select an arbitrary variation of ω along the σ_1 curve (figure 8a). Now perform a test in which the stress σ is applied for time t_1 when the stress is increased to σ_2 and maintained constant at this value to give the creep curve shown in figure 8b. Since a single state variable is sufficient to define the creep curves it follows that section 12 of the curve 012 must be identical in shape with a portion of the constant stress creep curve σ_2. In this way the state ω_1 may be defined on the σ_2 curve. By performing similar tests constant ω contours may be constructed as shown in figure 9. This information can then be used to predict the variation of strains and damage in a result of variable stress histories (Leckie & Hayhurst 1975).

The procedures described above are applicable provided that the state of damage can be described by a single damage parameter. If a single state parameter suffices it might still be necessary to modify the equations (4.1) for loading and unloading structures. This is certainly the case for creep deformations in the primary region when it is found that the growth laws of dislocation density differ for loading and unloading. This point has been discussed by Leckie & Ponter (1974). A systematic programme of step loadings for defining the growth laws of copper in the tertiary region is being performed by D. R. Hayhurst & C. J. Morrison (private communication). The test results indicate that more than one state variable is required with both dislocation density and damage being necessary state variables.

(b) *The Rabotnov–Kachanov equations*

Unfortunately it is difficult and time consuming to conduct experiments described in the previous section and the variable stress experiments which have been performed are limited to the prediction of rupture life under cyclic conditions of loading. Faced with this difficulty Rabotnov (1969) proposed modifications of constitutive equations first suggested by Kachanov (1958) which described existing experimental results. For uniaxial stress tests the growth laws are assumed to have the simple form

$$\dot{\epsilon}/\dot{\epsilon}_0 = (\sigma/\sigma_0)^n/(1-\omega)^m;$$
$$\dot{\omega}/\dot{\omega}_0 = (\sigma/\sigma_0)^\nu/(1-\omega)^\eta. \qquad (4.2\,a, b)$$

It is assumed that $\omega = 0$ when the material is in its undamaged state and $\omega = 1$ at rupture. In the equations, n, m, ν, η, $\dot{\epsilon}_0$, $\dot{\omega}_0$ and σ_0 are constants to be defined; η is defined below. It should be noted that in common with the spirit of § 4.1 no precise physical definition of the parameter ω is attempted. In fact this point has been specifically made by Rabotnov (1969) but for some reason there has been a persistent misrepresentation in some literature by interpreting the term $(1-\omega)$ as the current cross-sectional area and the term $\sigma/(1-\omega)$ as the effective stress.

When at the beginning of the test and the material is undamaged $\omega = 0$ and equation (4.2a) reduces to

$$\dot{\epsilon}/\dot{\epsilon}_0 = (\sigma/\sigma_0)^n, \qquad (4.3)$$

which is the usual form of the steady state creep equation. In this equation $\dot{\epsilon}_0$ is the steady-state strain rate corresponding to the applied stress σ_0 and the constant n is the stress index.

For constant stress it is easy to integrate the equations (4.2b) to give the time variation of strain and damage. By applying the rupture condition $\omega = 1$ it is also possible to determine the time to rupture t_R. Applying these conditions gives the following results:

$$\epsilon/\epsilon^* = \lambda[1-\{1-t/t_R\}^{1/\lambda}];$$
$$\omega = 1-(1-t/t_R)^{1/(\lambda+1)}, \qquad (4.4\,a, b)$$

where

$$t_R = 1/(\eta+1)\,\dot{\omega}_0(\sigma/\sigma_0)^\nu;$$
$$\epsilon^* = \dot{\epsilon}_0(\sigma/\sigma_0)^n\,t_R = \dot{\epsilon}_0(\sigma/\sigma_0)^{n-\nu}/(\eta+1)\,\dot{\omega}_0;$$
$$\lambda = (\eta+1)/(\eta+1-m). \qquad (4.4\,c\text{–}e)$$

From equation (4.4c) the rupture time t_0 corresponding to the applied stress σ_0 is

$$t_0 = 1/(\eta+1)\,\dot{\omega}_0,$$

and ϵ^* is equal to the initial strain rate times the time to rupture. At rupture the strain is $\lambda\epsilon^*$ and the strain/time curve has the form shown in figure 10. It should be noted in passing that n is normally greater than ν so that prediction of the theory is that the creep strain at rupture decreases with decreasing stress. This is in accord with experimental observations. The value of λ can be obtained from a measurement of the strain at rupture and it can be seen that the form of the dimensionless strain/time curve is dependent on the value of λ only. From the expression for the time to rupture and the test results it is possible to obtain the values of $(\eta+1)\,\dot{\omega}_0$ and ν.

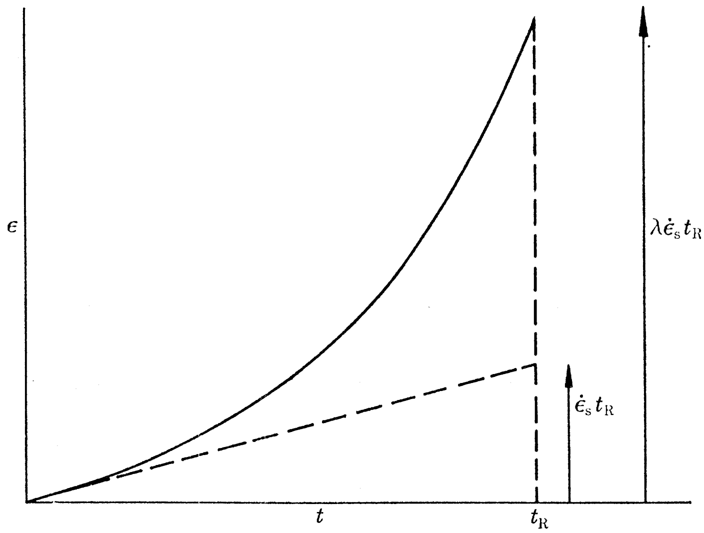

FIGURE 10. Determination of constant λ.

However, the information available is insufficient to pinpoint all the constants, which implies that there is an excess of constants available. If, for convenience, m is selected to be equal to the constant n the equations then become

$$\epsilon/\dot{\epsilon}_0 = \{\sigma/(1-\omega)\,\sigma_0\}^n;$$

$$\dot{\omega} = (\lambda-1)\,(\sigma/\sigma_0)^\nu/\lambda n t_0\,(1-\omega)^\eta, \qquad (4.5\,a, b)$$

where

$$\eta = \lambda n/(\lambda-1)-1.$$

The constants are now in a form which is convenient for comparison with experimental results.

(c) *Generalization of the Rabotnov–Kachanov equations*

Leckie & Hayhurst (1974) have attempted to generalize the Rabotnov–Kachanov equations to multiaxial states of stress. The form of the generalizations was influenced by the experimental results of Johnson *et al.* (1962) which indicate that the strain rates are dependent on the effective stress $\bar{\sigma}$ and that the components of strain rate $\dot{e}_{ij}$ are proportional to the deviatoric stress s_{ij}. These results when expressed mathematically take form

$$\dot{e}_{ij}/\dot{\epsilon}_0 = \tfrac{3}{2}(\bar{\sigma}/\sigma_0)^{n-1}\,s_{ij}/\sigma_0, \qquad (4.6)$$

with n, $\dot{\epsilon}_0$ and σ_0 having the same significance as in (4.5). Investigation of the torsion-tension experimental results of Johnson *et al.* on copper and aluminium show in both cases that, when deterioration is taking place in the tertiary region, the ratio of the strain rate components remains sensibly constant and equal to the value in the steady state condition. Consequently the strain rates in the tertiary region must be represented by constitutive equations with the form

$$\dot{\epsilon}_{ij}/\dot{\epsilon}_0 = \tfrac{3}{2}k(t)\,(\bar{\sigma}/\sigma_0)^{n-1}\,s_{ij}/\sigma_0, \tag{4.7}$$

where $k(t)$ is a scalar quantity increasing monotonically with time. Johnson *et al.* also observed that the growth of k is dependent on the multiaxial form of the stress field. In copper it is maximum stress dependent and in aluminium shear stress dependent. In fact the stress state affecting the growth of k appears to be the same as that dictating the form of the isochronous surface.

Comparing the form of equation (4.7) with the Rabotnov–Kachanov uniaxial relation (4.5*a*) suggests the following constitutive equation for the strain rates:

$$\dot{\epsilon}_{ij}/\dot{\epsilon}_0 = \tfrac{3}{2}(\bar{\sigma}/\sigma_0)^{n-1}\,(s_{ij}/\sigma_0)/(1-\omega)^n. \tag{4.8a}$$

The proposed damage rate equation following (4.5*b*) has the form

$$\dot{\omega} = \{(\lambda-1)/\lambda n t_0\}\,\Delta^{\nu}(\sigma_{ij}/\sigma_0)/(1-\omega)^{\eta}, \tag{4.8b}$$

where
$$\eta = \{\lambda n/(\lambda-1)\}-1,$$

and t_0 and ν have the same significance as in equations (4.5) and $\Delta(\sigma_{ij}/\sigma_0) = 1$ is the isochronous surface for failure time t_0.

Integrating this equation and using the condition $\omega = 1$ at the rupture time t_R gives

$$t_R/t_0 = 1/\Delta(\sigma_{ij}/\sigma_0)^{\nu}.$$

The damage rate equation reduces to the uniaxial form and satisfies the condition that the isochronous rupture surface for rupture time t_0 is $\Delta(\sigma_{ij}/\sigma_0) = 1$. The value of ω in equation (4.8*b*) is also a function of $\Delta(\sigma_{ij}/\sigma_0)$ thereby satisfying the condition that the growth of the scalar k in equation (4.7) is affected by the same stress state as that for rupture. The proposed constitutive equations appear to satisfy the macroscopic observations and are in a mathematical form which allows further exploitation in continuum mechanics.

5. The materials science approach

(*a*) *Introduction*

In this section an attempt is made to present, in a unified form, the studies of materials scientists and to identify features which are common to those of the phenomenological approach described above.

A general formulation for the growth of damage at grain boundaries has been given by Ashby & Raj (1975). The growth of damage is expressed in terms of two mechanisms. One mechanism, referred to as nucleation, gives a measure of the rate at which grain boundary voids are formed. The other mechanism, referred to as growth, gives a measure of the rate of growth of void size.

The nucleation rate is represented by $\mathrm{d}n/\mathrm{d}t$ and is measured in the number of voids formed per unit time on unit area of grain boundary. Holes apparently nucleate at points of strain concentration which can occur at inclusions contained in the grain boundary or at the extremities of slip bands within the crystal. The growth of holes can apparently be the result of the diffusion of vacancies along the grain boundary or of concentration of creep strains at inclusions.

Since voids are forming and growing continuously it is necessary to use the integral relation derived by Ashby & Raj to obtain a measure of the total damage. Suppose at time τ the nucleation rate is $\dot{n}(\tau)$ so that in time interval $d\tau$ the number of new voids formed is $\dot{n}(\tau)\,d\tau$. At a later time, t, these holes will be growing and the rate of increase of cross sectional area of the voids formed at time τ is $\dot{a}(t,\tau)$. During the time interval $(t-\tau)$ the holes formed at time τ will then have a cross sectional area

$$\dot{n}\,d\tau \int_{\tau}^{t} \dot{a}(t,\tau)\,dt.$$

The total area of the voids at time t_f is then

$$A(t_f) = \int_{\tau=0}^{\tau=t_f} \dot{n}(\tau)\,d\tau \int_{t=\tau}^{t=t_f} \dot{a}(t,\tau)\,dt. \tag{5.1}$$

Instead of using A as a measure of damage, some authors prefer to use the volume of the voids. Using the same method, illustrated above, the total volume V of the voids is

$$V(t_f) = \int_{\tau=0}^{\tau=t_f} \dot{n}(\tau)\,d\tau \int_{t=\tau}^{t=t_f} \dot{v}(t,\tau)\,dt, \tag{5.2}$$

where $\dot{v}$ is the volume growth rate of the voids.

The formulation is quite general but it implies that to calculate damage it is necessary to keep track of the size of each void as it is formed. This is equivalent to a multistate variable theory in which the number of state variables corresponds to the number of voids and is increasing with time. It is unlikely that the integral can be calculated in closed form except for simple loading histories or for special forms of the rate equations.

Various studies have been made with the objective of formulating the equations which define the nucleation and void growth rates. Two particular examples are now discussed against the background of the phenomenological procedures previously discussed.

(*b*) *The Greenwood growth equations*

Greenwood (1973) has studied the growth of creep damage in copper at a temperature of 500 °C. The tests were performed under conditions of constant uniaxial stress and measurements made of the number and size of voids. The visual observations indicate that the voids are all approximately the same size which implies that when a void is first formed it grows rapidly in size until it catches up with the voids formed earlier. It is possible to express this observation in a suitable mathematical form (Leckie & Hayhurst 1977) which can be substituted in equation (5.1), but it is easy to see that the total cross-sectional area is simply $A = na$. No attempt was made to study the effect of damage on strain rate although it appears that the tertiary strains were little bigger than those predicted by steady state theory.

Suppose that a uniaxial test is conducted at constant stress σ_0. Let the rupture time be t_0 when the strain is ϵ_0, the hole density n_0 and the average volume of the holes v_0. With reference to these physical values the growth equations for a uniaxial stress σ are

$$\begin{aligned} d(n/n_0)/dt &= (\sigma/\sigma_0)^2\, d(\epsilon/\dot{\epsilon}_0)/dt; \\ d(v/v_0)/dt &= (\sigma/\sigma_0)/t_0; \\ d(\epsilon/\epsilon_0)/dt &= (\sigma/\sigma_0)^5/t_0. \end{aligned} \tag{5.3a–c}$$

In the first of these equations the nucleation rate is proportional to the strain rate but is, in addition, proportional to the square of the applied stress. In the second the volume rate of the voids is assumed to be controlled by a diffusion process which is proportional to the stress. The third

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

equation illustrates that the strain rate is proportional to the fifth power of applied stress and the effect of damage on strain rate is neglected.

The damage is defined as

$$\omega = nv^{\frac{2}{3}}/n_0 v_0^{\frac{2}{3}} = A/A_0, \tag{5.3d}$$

and the rupture condition as $\omega = 1$. This is the same condition as proposed by Greenwood (1973) who used the concept of a critical area fraction at failure. For multiaxial states of stress Leckie & Hayhurst (1977) suggested that equations (5.3*a*–*d*) take the form

$$\mathrm{d}(n/n_0)/\mathrm{d}t = (\sigma_1/\sigma_0)^2\,\mathrm{d}(\bar{\epsilon}/\epsilon_0)/\mathrm{d}t;$$

$$\mathrm{d}(v/v_0)/\mathrm{d}t = (\sigma_1/\sigma_0)/t_0;$$

$$\mathrm{d}(\bar{\epsilon}/\epsilon_0)/\mathrm{d}t = (\bar{\sigma}/\sigma_0)^5/t_0;$$

and the damage is

$$\omega = nv^{\frac{2}{3}}/n_0 v_0^{\frac{2}{3}}. \tag{5.4a–d}$$

Integrating the above conditions and applying the rupture condition $\omega = 1$ gives the isochronous surface for rupture time t_0:

$$\Delta(\sigma_{ij}/\sigma_0) = (\sigma_1/\sigma_0)^{\frac{8}{23}}(\bar{\sigma}/\sigma_0)^{\frac{15}{23}} = 1. \tag{5.5a}$$

Leckie & Hayhurst (1977) have shown that for proportional loading (i.e. stress histories for which the ratio of component stresses remain constant) the equations (5.4*a*–*d*) can, by introducing certain small modifications, be expressed in the form

$$\mathrm{d}(\epsilon/\epsilon_0)/\mathrm{d}t = (\bar{\sigma}/\sigma_0)^5/t_0,$$

$$\mathrm{d}\omega/\mathrm{d}t = \tfrac{2}{3}\Delta^{\frac{23}{5}}(\sigma_{ij}/\sigma_0)/(1-\omega)^{\frac{1}{2}}, \tag{5.5b, c}$$

with the rupture condition $\omega = 1$. These equations are a special case of the Kachanov–Rabotnov type equations (4.8) in which the damage state variable ω is equal to the normalized form of the damage relation of equation (5.1).

(*c*) *The Dyson–McLean equations*

Dyson & McLean (1977) have performed constant uniaxial tension and torsion tests on Nimonic 80A at 700 °C. In addition to measuring tertiary strains and times to rupture, measurements were also made of the number of voids per unit area of grain boundary and of the total volume of voids.

For constant uniaxial stress tests carried out at stress σ_0, the rupture time is t_0 at which time the void density is n_0 and the total volume of the voids is V_0. The constitutive equations proposed by Dyson & McLean (1977) give expressions for the rate of void nucleation and of void growth. Damage is expressed as the total volume of voids and the effect of damage on the creep rate is also included. For multiaxial states of stress the proposed rate equations may be written in the form

$$\mathrm{d}(\bar{\epsilon}/\epsilon_0)/\mathrm{d}t = 3(\bar{\sigma}/\sigma_0)^4(v/v_0)^{\frac{4}{9}}/t_0;$$

$$\mathrm{d}(n/n_0)/\mathrm{d}t = \tfrac{1}{2}(\sigma_1/\bar{\sigma})^2(\epsilon_0/\bar{\epsilon})^{\frac{1}{2}}\,\mathrm{d}(\epsilon/\epsilon_0)/\mathrm{d}t;$$

$$\mathrm{d}(v/v_0)/\mathrm{d}t = (\sigma_1/\bar{\sigma})^{0\cdot 7}\,\mathrm{d}(\epsilon/\epsilon_0)/\mathrm{d}t; \tag{5.6a–c}$$

where

$$v_0 = 3V_0/2n_0.$$

The damage is defined in terms of the total volume of voids which are defined by the integral relation of equation (5.2). Then

$$\omega = \frac{V}{V_0} = \frac{3}{4}\int_{\epsilon_1=0}^{\epsilon_1=\epsilon} (\sigma_1/\bar{\sigma})^2\,(\epsilon_0/\epsilon_1)\,\mathrm{d}(\epsilon_1/\epsilon_0)\int_{\epsilon_1}^{\epsilon}(\sigma_1/\sigma)^{0.7}\,\mathrm{d}(\epsilon/\epsilon_0).$$

For the case of proportional loading Leckie & Hayhurst (1977) were able to show that the above equations may, with small modification, be reduced to the following growth equations

$$\mathrm{d}(\epsilon/\epsilon_0)/\mathrm{d}t = \{\bar{\sigma}/\sigma_0/(1-\omega)\}^4;$$

$$\mathrm{d}\omega/\mathrm{d}t = \Delta^4(\sigma_{ij}/\sigma_0)/(1-\omega)^4;$$

where $$\Delta(\sigma_{ij}/\sigma_0) = [\sigma_1^{1.8}\bar{\sigma}^{2.2}/\sigma_0^4]^{\frac{1}{4}},$$

with $\Delta(\sigma_{ij}/\sigma_0) = 1$ defining the isochronous surface for rupture time t_0. The rupture condition is again $\omega = 1$.

(d) *Some observations*

It would appear that for proportional stress† histories the phenomenological constitutive equations developed in § 4 using a single state damage variable satisfy the macroscopic observations and are also capable of physical interpretation. However, for non-proportional loading it appears necessary to provide growth equations for both nucleation and void growth and in these circumstances at least two state variables will be required. Fortunately, in many practical components, while stresses vary they do remain sensibly proportional. Nevertheless, problems will arise in which the stress fields have large rotations when the sequence of loading is applied. Growth and nucleation will occur at different rates in particular directions and it is to be expected therefore that the description of damage will be tensorial in form. It will also be necessary to understand how and if damage introduces important anisotropic effects. An experimental investigation into this problem has been started by Hayhurst & Morrison (1977). In these experiments plates have been subjected to plane stresses in the ratio 2:1 and after times in excess of half life the stress field is rotated to give the ratio 1:2. If the material deforms according to a Mises law then no deformation should be observed in the direction of the lower stress in a 2:1 ratio test. It is found that this rule is obeyed even after the stress rotation which suggests that it is a scalar form of the damage tensor which affects the value of the strain rate. It is not surprising that this result should hold for aluminium since damage and strain appear to be related, but it is more surprising in the case of copper for which damage occurs in the direction of maximum stress. Once this test programme has been completed it should be possible to formulate more complete constitutive laws which can describe the effect of large rotations of stress.

6. Applications of the constitutive laws

By using the constitutive equations (4.8) in conjunction with the continuum conditions of equilibrium and compatibility it is possible to calculate the time-dependent stress and strain fields in load bearing components. It is also possible to calculate the growth of damage so that the time to first local failure and final rupture may be determined. Apart from problems of especially simple geometry it is normally necessary to resort to the use of the finite element method. An example illustrating the procedure is that solved by Hayhurst, Dimmer & Chernuka (1975) of the plate under tension penetrated by a hole. The results of the calculations indicate

† I.e. a stress régime in which the applied forces maintain their direction while varying in magnitude.

that high stresses exist at the edge of the hole during the early part of the component life. However, as time progresses damage develops in the vicinity of the hole with the result that the high stresses are relaxed and equilibrium is maintained by an increase of stress in the less severely loaded regions. This stress redistribution is beneficial in extending the life of the component, but it makes it difficult to estimate the remaining life of a component on the basis of physical measurement of the damage as suggested by Dyson & McLean (1972). Patterns of damage calculated by Hayhurst *et al.* (1975) are plotted in figure 1. and compare well with the damage observed in a uniaxial plate made from copper. In order to maintain numerical stability in the calculations it is necessary to use very small time step increments and, as a result, even for this simple component heavy demands are made on the IBM 370/195 computer. There is therefore a strong motivation to develop approximate methods which give results good enough for design purposes and which avoid the need for demanding and detailed analysis. A complete analysis often provides more information than is required and the calculations often obscure rather than highlight those features which dominate component performance. It will be illustrated how approximate calculations isolate the reference rupture stress as the material parameter which most strongly influences component life.

(*a*) *Upper bound calculations*

Upper bound calculations of rupture life can be obtained by making the assumption that the isochronous surface $\Delta(\sigma_{ij}/\sigma_0) = 1$ is convex. The experimental evidence presented by Hayhurst (1972) suggests that in many circumstances this is likely to be a justified assumption. The assumption appears always to be valid when the compressive stress is smaller in magnitude than the tensile stress. In aluminium the isochronous surface is effective stress dependent even when the stresses are compressive so that the surface is always convex. However, in copper the isochronous surface according to equation (5.5*a*) is concave when the magnitude of the compressive stress is somewhat greater than the tensile stress. More testing is required in the region of compressive stress space before the isochronous surface for copper can be fixed.

Suppose a structure of total volume V is subjected to a set of constant loads P_i. An upper bound t_u on the rupture life t_R of the structure is given by the following expression (Leckie & Wojewódzki 1975)

$$t_R/t_0 < t_u/t_0 = V\Big/\int_V \Delta^\nu(\sigma^s_{ij}/\sigma_0)\,dv. \tag{6.1}$$

In this expression σ^s_{ij} is the stress distribution found by performing the analysis on a structure of identical geometry and load to that under consideration but with a creep law of the form

$$\dot{\epsilon}_{ij}/\dot{\epsilon}_0 = \Delta^{\nu-1}(\sigma_{ij}/\sigma_0)\,\partial\Delta/\partial(\sigma_{ij}/\sigma_0). \tag{6.2}$$

This calculation is equivalent to the standard steady state analysis but with a different creep law. The substitution of the stress field σ^s_{ij} into equation (6.1) is a simple calculation.

It is found convenient in practice to express the result (6.1) in terms of the so-called reference rupture stress. If a uniaxial specimen is subjected to a constant stress σ_u so that the rupture life is t_u then

$$t_u/t_0 = (\sigma_0/\sigma_u)^\nu.$$

Equating this to the time t_u calculated for the component gives an expression for σ_u according to the expression

$$\frac{\sigma_u}{\sigma_0} = \left\{\frac{1}{V}\int_V \Delta^\nu(\sigma^s_{ij}/\sigma_0)\,dV\right\}^{1/\nu}. \tag{6.3}$$

If this value σ_u is used in conjunction with the uniaxial creep rupture data then an upper bound is obtained on the time to rupture of the structure. This is a convenient way of relating component performance directly to uniaxial data and avoids data fitting procedures. The stress is known as the *reference rupture stress*.

Another upper bound on rupture time has been developed by Goodall, Cockroft & Chubb (1975). In their calculation the limit load P_0 of a component of identical geometry is determined for a material with a yield stress σ_0 and a yield surface

$$\Delta(\sigma_{ij}/\sigma_0) = 1,$$

which is identical in shape to the isochronous surface. Then the representative rupture stress σ_u is given by

$$\sigma_u/\sigma_0 = (P_i/P_0), \tag{6.4}$$

and once again when used in conjunction with the uniaxial creep rupture data gives an over-estimate of the rupture life of the component. This expression gives a better bound than that of (6.3) but the difference is not great and the selection of the procedure will normally be related to convenience and local computing expertise.

The expression (6.3) illustrates that the reference rupture stress is related to a weighted average stress over the volume which is in accord with the experimental observations discussed in § 1. While the above procedures do give upper and therefore unsafe bounds on rupture life, experiences reported in the quoted references suggest that the bounds are often good enough for practical design purposes. However, more experience about the effectiveness of the bounds is still required.

(*b*) *Lower bound calculations*

It has proved difficult to find a reference rupture stress which, when used in conjunction with the uniaxial data, gives a lower bound on rupture life. Some limited progress has been made for components which are kinematically determinate (in contrast to a statically determinate system in which all of the stresses can be expressed in terms of the loads by means of equilibrium considerations, a kinematically determinate structure is one for which all strains can be expressed in terms of a discrete displacement parameter using the conditions of compatibility). For materials for which the isochronous surface is given by the Mises condition $\Delta(\sigma_{ij}/\sigma_0) = (\bar{\sigma}/\sigma_0)$ then the reference rupture stress σ_L found by Leckie & Hayhurst (1974) giving a lower bound on time is

$$\sigma_L/\sigma_0 = \left\{\int_V (\bar{\sigma}_s/\sigma_0)^{n+1+\nu}\,dv \Big/ \int_V (\bar{\sigma}_s/\sigma_0)^{n+1}\,dv\right\}^{1/\nu}, \tag{6.5}$$

where $\bar{\sigma}_s$ is the steady state effective stress distribution.

For components in a state of plane stress the reference rupture stress is

$$\sigma_L/\sigma_0 = \left\{\int_V (\bar{\sigma}_s/\sigma_0)^{n+1} \Delta^\nu(\sigma^s_{ij}/\sigma_0)\,dv \Big/ \int_V (\bar{\sigma}_s/\sigma_0)^{n+1}\,dv\right\}^{1/\nu}, \tag{6.6}$$

where σ^s_{ij} is the steady state stress distribution.

Again it can be observed that this reference stress, which gives a lower or conservative estimate of rupture life, is a weighted average of stresses over the component volume, and can be calculated using steady state stress distributions.

(c) *Examples*

The formulae which have been developed for the reference rupture stress require stress distributions which may be obtained using steady state creep or limit load analysis, both of which are standard engineering calculations. With these stress distributions available, the calculations of the integrals is straightforward and avoids the computationally demanding task of integrating forward in time. The results of the steady state analysis of a plate in tension pierced by a hole were substituted in equation (6.6) (Leckie & Hayhurst 1974) to give a reference rupture stress which is only 1 % greater than the average stress at the minimum cross section. This result is in accord with the experimental result discussed in § 1 which indicated that the rupture life of the plate was dependent on the average stress at the minimum cross section. The calculations were very much less demanding than those required for the full analysis reported by Hayhurst *et al.* (1975).

Table 2. Reference rupture stress for cylinder with internal pressure

$\sigma_R/2p$	0.742	0.730	0.725
n	3	5	7

Apart from their computational utility perhaps the real advantage of the expressions for reference rupture stress is that they provide physical insight so that certain results can be anticipated.

Provided local stresses are kept to the level expected in practical design (for example elastic stress concentration factors are limited to 2.25 in certain pressure vessel components) it is found in steady state calculations that the value of $(\bar{\sigma}/\sigma_0)^{n+1}$ does not vary greatly throughout the component. Assuming this to be the case (equation (6.5) for example) gives the result that the reference rupture stress is independent of the values of the constants n and ν. In practice the value of $(\bar{\sigma}/\sigma_0)^{n+1}$ does vary so that the reference stress must be dependent on the values of ν and n, but it is to be expected that the dependence is not very strong.

The calculation of the reference stress σ_R for a cylinder with $\Delta(\sigma_{ij}/\sigma_0) = \bar{\sigma}/\sigma_0$ subjected to internal pressure p has been shown (Leckie & Hayhurst 1974) to be

$$\sigma_R/2p = \{[1-(a/b)^{2(1+\nu)/n}]/(1+\nu)\,[1-(a/b)^{(1+\nu)}]\}^{1/\nu}/n,$$

where a is the internal and b the external radius. If the value of ν is taken to be $0.7n$ (Odqvist 1974) then for the ratio $b:a = 2$ the values of the reference stress for different values of n are those shown in table 2. A practical expression for this component might then be

$$\sigma_R/p = 1.5, \tag{6.7}$$

which is independent of n and ν. The aim of most engineering calculations is to use simple expressions of this type which are particularly helpful in the initial stages of design. This example also illustrates how the reference stress σ_R may be regarded as a material parameter. If the required component life is t_R then σ_R is obtained from the uniaxial creep rupture data and the pressure p which the component can contain for time t_R can then be determined using equation (6.7). The design life of many components operating at temperature is 10^5 h and referring for example to pages 352 and 353 of the *British Steelmakers Creep Committee high temperature data*, the reference rupture stresses for different temperatures of a 1 % Cr $\frac{1}{2}$ %Mo alloy steel are those given in table 3. It is suggested that the representative rupture stress is a convenient material property playing a rôle similar to that of yield stress so that for a given temperature only one value is required and the complex and bulky data appearing in the literature are greatly condensed. In performing the calculations relating component performance and material properties sensible values of ν and n ought to be selected but there seems little advantage to be gained in attempting to find precise values.

The formulae for representative stress have been determined for a number of components and the predictions compared with the results of tests on the components (Leckie & Hayhurst 1974). The comparisons are favourable, and the often dramatic effects resulting from the shape of the isochronous surface are verified. For example, when the formulae are applied to the notched cylindrical bar described in § 2 it is found that the reference stress for copper bars is approximately equal to the average stress at the minimum section, while for aluminium bars the reference stress is considerably smaller than the average stress and accurately predicts the strengthening observed in experiment.

Table 3. Reference rupture stresses for an alloy steel

$\sigma_R/(\text{kgf/mm}^2)$	30	15	5.5
$T/°\text{C}$	450	500	550

In order to complete the calculation for the reference rupture stress it is necessary to know the form of the isochronous surface. This information may not always be available and it is then necessary to assume a shape which gives the most conservative results. For example, there are many engineering components which operate in a state of plane stress and it is then conservative to assume that the isochronous surface is defined by the effective stress, i.e. $\Delta(\sigma_{ij}/\sigma_0) = (\bar{\sigma}/\sigma_0)$. It is then not too difficult to find a representative stress. If the classical limit load of a component is P_0 for a yield stress σ_0 then the reference representative stress when the applied load is P_i is, from equation (6.4),

$$\sigma_R/\sigma_0 = P_i/P_0 = \lambda. \tag{6.8}$$

This approximation can be used to advantage in the design of reinforcement of pressure vessel intersections. The cylinder/sphere intersection for example is normally designed by providing reinforcement (figure 11, plate 2) so that the limit load of the reinforcing pad is equal to the limit load of the membrane. In this way the pressure vessel is of uniform strength. If the intersection designed according to this procedure operates at temperature then the value of λ in equation (6.8) is the same for the membrane and the reinforcement so that the failure should apparently occur in both simultaneously. However, a further modification must be made to the calculations to account for the effect of geometry change resulting from creep deformation. It is known that as the intersection deforms its limit load increases while that of the membrane decreases. Consequently, the value of λ continuously decreases for the reinforcement and increases for the membrane (i.e. the reference stress decreases for the reinforcement and increases for the membrane). Hence the intersection becomes stronger than the membrane and consequently it is the membrane which should rupture first. The results of experiment verify this prediction (figure 12, plate 2), and the example illustrates that the behaviour of the intersection can be determined without the need of any complex calculations.

7. Conclusions

Some progress has been made towards formulating constitutive equations which can be used in a continuum mechanics approach to determine the growth of material damage in load bearing components operating in the creep range. It is suggested that the single damage parameter used in the phenomenological constitutive equations has a rather precise physical interpretation expressed in terms of an integral of the nucleation and growth of voids. The constitutive equations are limited, however, to proportional loading. For non-proportional loading the physical studies suggest that at least two state variables will be necessary in the constitutive equations.

In principle the constitutive equations can be used together with the continuum equations to calculate precisely the growth of damage in components, but the calculations are very demanding. Another approach has been to develop approximate methods and results are obtained which are much simpler to apply. The formulae indicate that it is average rather than point values which dictate failure life, and this is consistent with experimental observation. When applying the approximate methods it becomes evident that it is the reference rupture stress which is an important material parameter, and that other constants in the constitutive equations play only a minor rôle in influencing performance.

To date, studies have been limited to constant and proportional loading and attempts must now be made to extend the understanding of the effects of non-proportional loading especially those involving stresses induced by fluctuations in temperature. An increase of local temperature is normally accompanied by a decrease of stress and vice versa with a decrease in local temperature. There is limited understanding of this problem and few test results available to provide the necessary guidance. The study of the effectiveness of the procedures described has been largely limited to structures with small values of stress concentration factors. It remains to be seen if the methods are reliable for conditions involving high stress concentrations and in particular when these concentrations appear as the result of cracks and flaws in the component.

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Discussion

S. A. F. Murrell (*Department of Geology, University College London, Gower Street, London, WC1E 6BT*). Professor Leckie has asked whether creep rupture and the growth of voids by creep were processes which could occur in the depths of the Earth, where matter is subjected to high pressure. Geologists and geophysicists would answer that they do envisage processes in the Earth in which voids may play a rôle. At depths down to levels at which hydrous minerals can still survive as stable phases there is the possibility of such minerals decomposing and releasing water into pores in the rock (Murrell & Ismail 1976*a*). Carbon dioxide may also exist in pores in mantle rocks. In addition, partial melting would create voids containing melt under high pressure (Murrell & Ismail 1976*b*). Magmas are generated in the upper parts of the mantle, and earthquakes occur at depths down to 700 km. In so far as earthquakes and the movement of magma involve faulting processes it seems likely that voids have a rôle in them.

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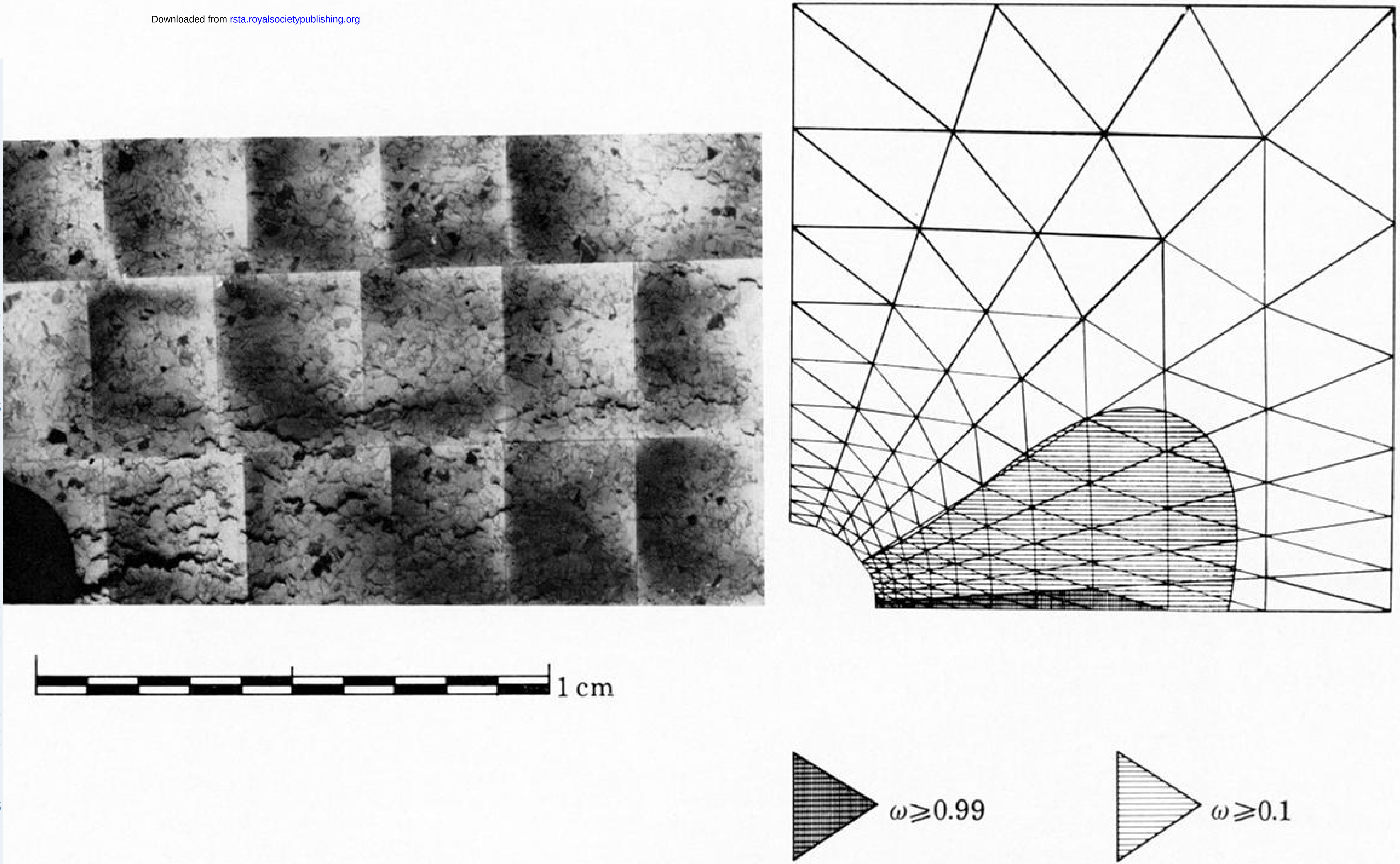

FIGURE 1. Rupture of copper tension panel with hole.

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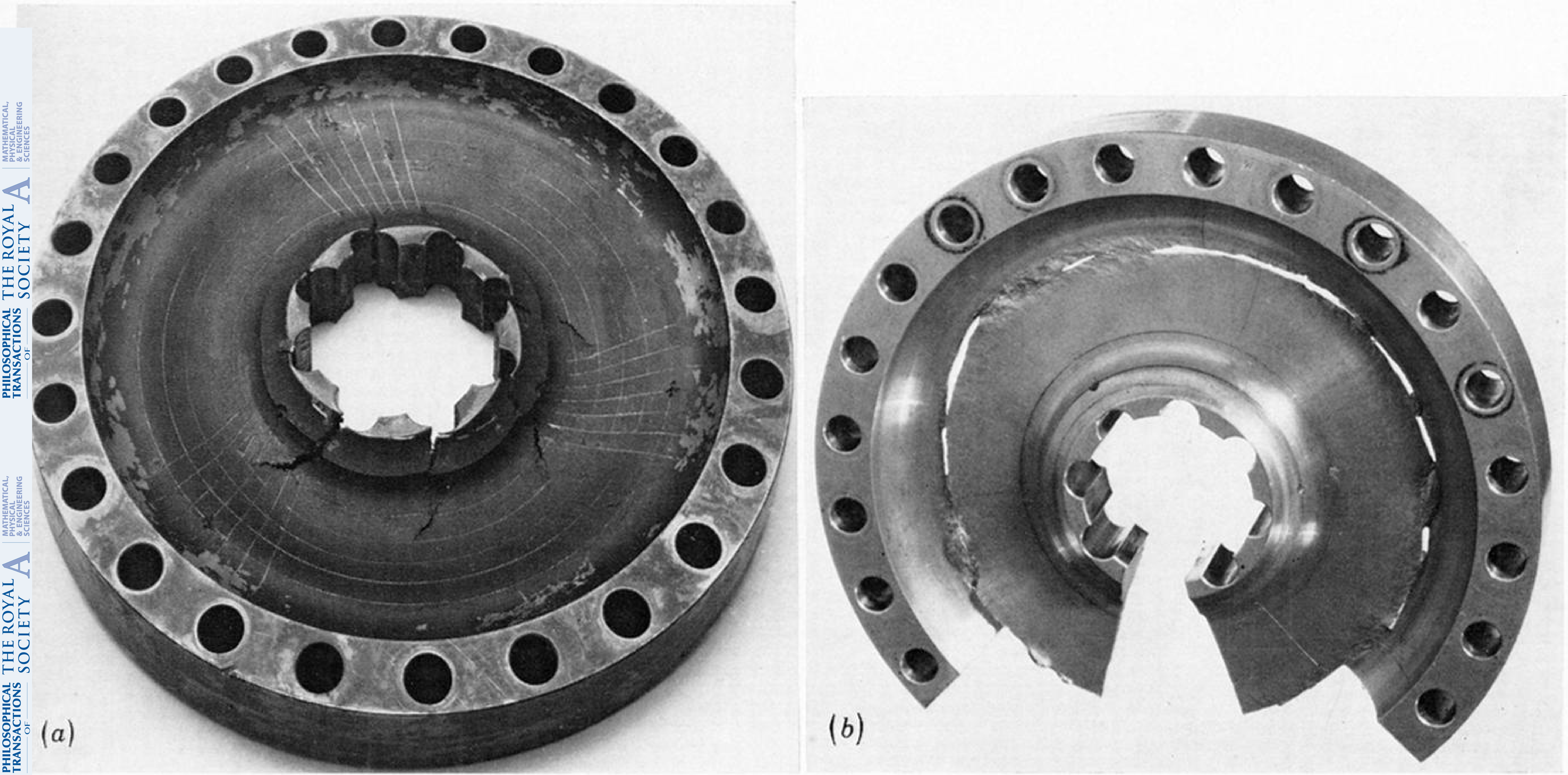
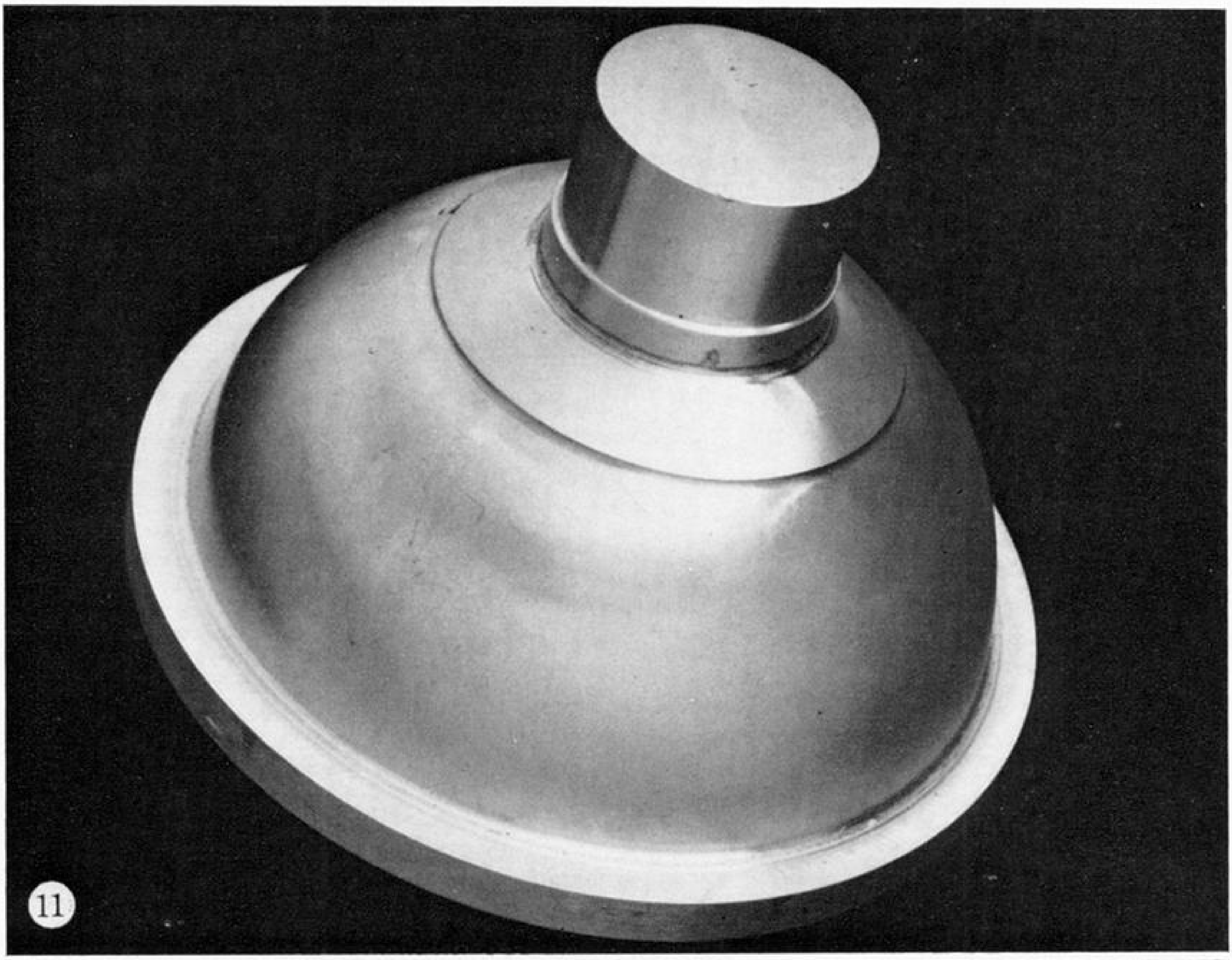


FIGURE 4. Andrade disk (*a*) copper (*b*) aluminium.



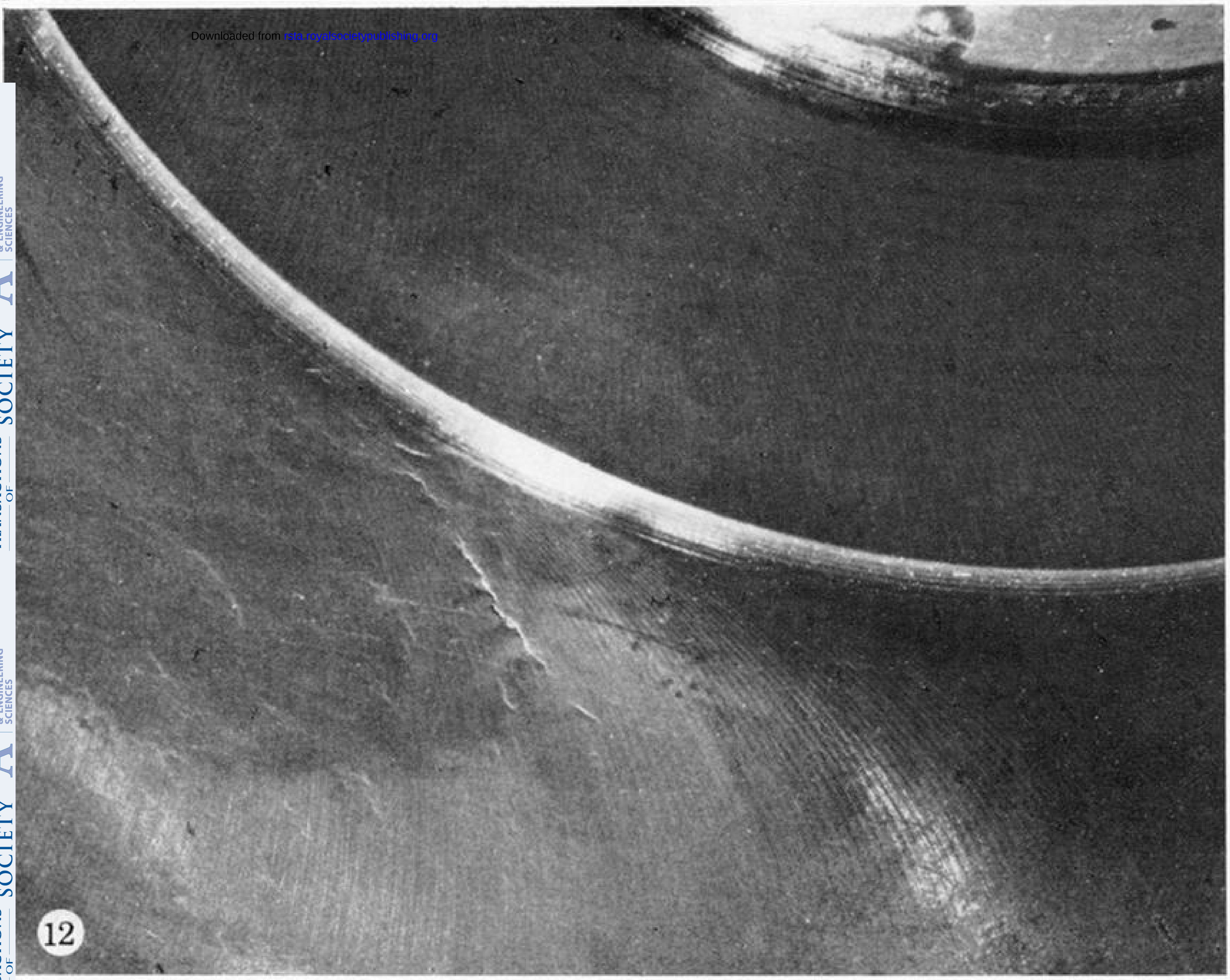


FIGURE 11. Balanced reinforced nozzle.
FIGURE 12. Rupture failure of membrane.